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Energy Release Rates for a Plane Crack Subjected to General Loading and their Relation to Stress-Intensity Factors

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Abstract
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Actually, J is the component of a vector in the plane of the crack and there exists a component of this vector normal to the crack plane, which, however, has not been interpreted properly in the past. It is one aim of this paper to supply a valid interpretation of this path-independent integral and to relate it to still another integral, also path-independent, which has been termed the L integral. It will be further shown explicitly that for a crack under mixed-mode loading this latter integral represents the energy release rate for rotation which can be used to determine both K_I and K_{II} .

Keywords
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ENERGY RELEASE RATES FOR A PLANE CRACK SUBJECTED TO GENERAL LOADING AND THEIR RELATION TO STRESS-INTENSITY FACTORS

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ABSTRACT

The well-known J integral of elastic fracture mechanics has been related to potential energy-release rate associated with crack extension and has proved to be of great value in fracture testing. In particular, the path-independence of the J integral has been used to an advantage in performing acoustoelastic measurements along a closed contour surrounding a crack tip.[8] In Mode I (opening mode) for example, the J integral depends essentially only on the corresponding stress intensity factor K_I which can thus be determined.

Actually, J is the component of a vector in the plane of the crack and there exists a component of this vector normal to the crack plane, which, however, has not been interpreted properly in the past. It is one aim of this paper to supply a valid interpretation of this path-independent integral and to relate it to still another integral, also path-independent, which has been termed the L integral. It will be further shown explicitly that for a crack under mixed-mode loading this latter integral represents the energy release rate for rotation which can be used to determine both K_I and K_{II}.

INTRODUCTION

For some years now, researchers became interested in path-independent integrals J, L and M. In the context of fracture mechanics the importance of path-independent integrals resides in the fact that they can be related to energy release rates, e.g. to crack extension forces which themselves depend only on stress intensity factors. It was shown by Rice[1] that the J integral is related to the elastic energy-release rate associated with statically extending cracks. Freund[2] has found that in certain special cases the M integral can be related to J, which he has shown to be useful in calculating stress intensity factors without solving the corresponding boundary value problem. The J integral is actually the component J_1 of a vector J_k, (k = 1,2), and since J represents a force, we could expect that J_2 is simply another component of that force. The relation between J_1 and M suggests that there might be a similar relation between J_2 and L, another path-independent integral studied by Knowles and Sternberg[3] and Rice and Budiansky.[4] In order to assign a practical use to J_2 or L (or both), we have to understand the physical meaning of those quantities and to establish possible relations between them (if they exist). This is the aim of this paper.

IS J_2 A PATH-INDEPENDENT INTEGRAL?

We consider a two-dimensional deformation field referred to Cartesian coordinates X_1, X_2. The crack of length 2a is placed along the OX_1 axis (see Fig. 1). The J integral is defined as

$$J = \oint_C (W dX_2 - T_i u_{i,1} d\ell) \tag{1}$$

where C is a contour enclosing the right crack tip, W the strain energy density, T_i the traction, u_i the displacement vector and comma denotes differentiation, e.g. $u_{i,1} = \partial u_i/\partial X_1$. If the crack is subjected to far-field homogeneous applied stresses σ_{11}^A, σ_{12}^A, σ_{22}^A, J equals to (in plane stress)

$$J = (K_I^2 + K_{II}^2)/E \tag{2}$$

where E is Young's modulus and K_I and K_{II} are stress intensity factors defined as

$$K_I = \sqrt{\pi a}\, \sigma_{22}^A \quad , \quad K_{II} = \sqrt{\pi a}\, \sigma_{12}^A . \tag{3}$$

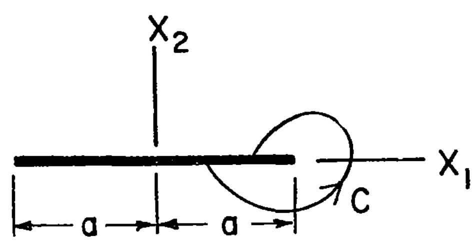

Figure 1

Path C for J Integral

We notice that J given by (2) does not depend on σ_{11}^A and is path-independent: because J can be expressed in terms of quantities which do not depend on the contour.

Actually, J is a component J_1 of a vector

$$J_k = \oint_C (W n_k - T_i u_{i,k}) d\ell \qquad i,k = 1,2 . \tag{4}$$

The second component of this vector J_2 has been calculated for the infinitesimal contour enclosing the crack tip giving

$$J_2 = -2K_I K_{II}/E \tag{5}$$

where again K_I , K_{II} are given by (3). Again J_2 does not depend on σ_{11}^A .

We observe an interesting feature of J_1 and J_2. As given by (4), they seem to be the components of a vector; however, comparing (2) and (5), they do not behave like independent quantities: if $J_1 = 0$, then J_2 is necessarily zero also, by contrast to the basic definition of a vector whose components should be independent from one another. We will return to this point later.

Now we would like to concentrate on the path-independence of J_1 and J_2 . The basic practical use of J_1 resides in the fact that it is path-independent, or more precisely: if a contour C encloses the crack tip and starts and ends on the crack face, the value of J_1 does not depend on the particular choice of C . The reason is that along the crack faces, T_i is zero, thus the second term of (1) is identically zero, while the first one vanishes because dX_2 is zero. This argument, however, does not hold for J_2: i.e. the second term (with T_i) vanishes, but not the first one, and J_2 is path-dependent in the same sense as J_1 is path-independent

Exact calculations are given for remote homogeneous fields in Ref. 5. Does this mean that J_2 has no meaning whatsoever? In order to answer this question, we have to establish more precisely what we mean by path-independence: J_1 strictly speaking is not path-independent, either, because if a contour encloses the whole crack, $J_1 = 0$ (see Fig. 2).

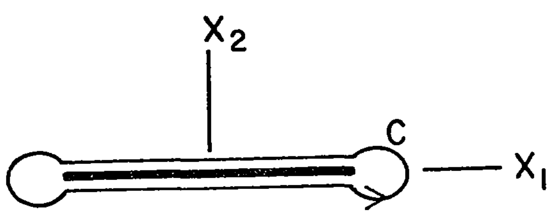

Figure 2

Path C for L Integral

Should a contour enclose a crack tip or the whole crack? Usually we could expect path-independence of some quantity if the contour is taken over all singularities. E.g., Gauss law in electro-statics is

$$\operatorname{div} \vec{E} = 4\pi\rho$$

where $\vec{E}$ denotes the electric field , ρ the charge density. In global form this law is stated as

$$\int_S \vec{E} \cdot d\vec{S} = 4\pi e$$

The surface integral of $\vec{E}$ gives the total charge e inside the volume and is independent on the surface S , as long as all charges are inside S (see Fig. 3).

$$\int_{S_1} \vec{E} \cdot d\vec{S} = \int_{S_2} \vec{E} \cdot d\vec{S} = 4\pi \sum_{i=1}^{n} e_i$$

Thus we have to decide whether a crack or a crack tip is a singularity under consideration. Actually, for J_2 there is no choice; only when the integral is taken around the whole crack does the contribution along the crack vanish and the total value of J_2 is zero (Ref. 5).

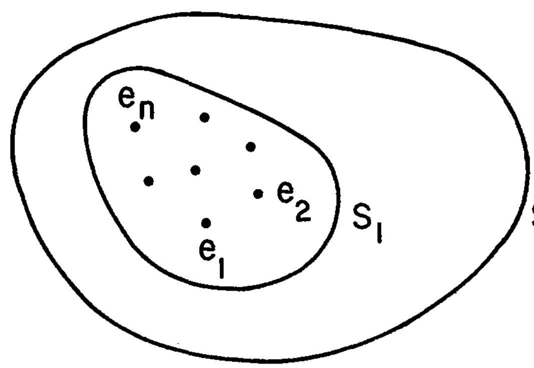

Figure 3

Charges Enclosed by Surfaces S_1 and S_2

RELATIONS BETWEEN M AND J RE-EXAMINED

Freund[2] established a relation between two path-independent integrals M and J . M is given by

$$M = \oint_C (W X_i n_i - T_k u_{k,i} X_i) dS \qquad (6)$$

and J is given by (1). In (6) the contour, however, is taken around the whole crack (see Fig. 2). The relation established by Freund is

$$M = 2aJ \qquad (7)$$

In other words, one quantity connected with the whole crack M is related to another one, namely J , associated with a crack tip only. This was the basis of some of the experimental work reported by R. King, G. Herrmann and G. Kino in a separate paper in these Proceedings.

Before we attempt establishing some relations between J_2 and another path-independent integral, let us re-examine the relation between M and J from the physical point of view. As it is well known, J is related to energy release rates. Let us start with the strain energy of an elastic continuum without defects

$$W = W[\varepsilon_{ij}(X_k)] \tag{8}$$

where $\varepsilon_{ij}(X_k)$ is the strain tensor. The body is in equilibrium, thus

$$\sigma_{ij,j} = 0 \tag{9}$$

where σ_{ij} is the symmetric stress tensor related to W

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \cdot \tag{10}$$

If we differentiate W with respect to X_i, and make use of (10), of the symmetry of stress tensor and of (9) we obtain

$$\begin{aligned} \frac{\partial W}{\partial X_i} &= \frac{\partial W}{\partial \varepsilon_{jk}} \varepsilon_{jk,i} = \sigma_{jk} u_{j,ki} = \sigma_{jk} u_{j,ik} \\ &= (\sigma_{jk} u_{j,i})_{,k} - \sigma_{jk,k} u_{j,i} = (\sigma_{jk} u_{j,i})_{,k} \end{aligned} \tag{11}$$

and finally

$$\frac{\partial}{\partial X_k} (W\delta_{ik} - \sigma_{jk} u_{j,i}) = 0 \cdot \tag{11}$$

If we integrate this expression over any volume inside the body, and then make use of Gauss theorem, we have

$$\int_S (Wn_i - T_j u_{j,i}) dS = 0 \cdot \tag{12}$$

For the two-dimensional case S becomes a line and (12) represents a conservation law. Our considerations concerned energy, but (12) is not conservation of energy. Actually, if we define the material momentum tensor b_{ik} as

$$b_{ik} = W\delta_{ik} - \sigma_{jk} u_{j,i} \tag{13}$$

relation (11) can be rewritten as

$$b_{ik,k} = 0 \tag{14}$$

which is very similar to the equation of equilibrium (9), expressing absence of body forces or conservation of linear momentum. In this sense (14) expresses conservation of material momentum in the absence of material forces. To reinforce this suggestion, let us observe that the integral in (12) is identical to that of J_k given by (4); however, the right-hand side of (12) equals zero. If we assume now, in order to distinguish between a continuum without and with defects, that W depends on X_i not only through ε_{ij}, but also explicitly, i.e.,

$$W = W[\varepsilon_{ij}(X_k), X_i]$$

we obtain

$$\frac{\partial W}{\partial X_i} = \frac{\partial W}{\partial \varepsilon_{jk}} \varepsilon_{jk,i} + \left(\frac{\partial W}{\partial X_i}\right)_{exp}$$

which leads to

$$b_{ik,k} = \left(\frac{\partial W}{\partial X_i}\right)_{exp} \tag{15}$$

instead of (14). The right-hand side represents a density of material force, in the same sense as f_i given by

$$f_i = \sigma_{ik,k} \tag{16}$$

represents a body force density; actually if f_i is a gravity force, f_i can be written as

$$f_i = -\frac{\partial U}{\partial x_i}$$

where U is the density of the gravitational potential, to make the formal analogy even stronger.

In classical mechanics, there is a strong relation between admissible transformations and conservation laws: e.g., translational invariance is connected with momentum conservation; presence of forces expresses non-conservation of momentum and in this sense is related to translations. So actually we can "deduce" the forces acting on the object under consideration by inducing a possible translation of that object. In this sense J is precisely a force acting on a crack tip, because to derive it the possible translation of a crack tip, not of a crack, was considered.

Another transformation of interest can be similarity. For a particular case of a crack it can be written as

$$\delta X_1 = \frac{\alpha}{a} X_1 \tag{16}$$

where α is an infinitesimal parameter. This transformation concerns X_1 only if we use the coordinate system of Fig. 1. This transformation leads to the formula (6) for M. The physical meaning of J_1 and M is clear: the translation of both tips by α along X_1, but in opposite directions (Fig. 4a), so $\delta X_1 = \alpha$ at the right tip, and $\delta X_1 = -\alpha$ at the left one, is equivalent to the similarity transformation given by (16) and characterized by the same α (Fig. 4b). In this sense M represents a generalized extension force acting on a crack (Fig. 4).

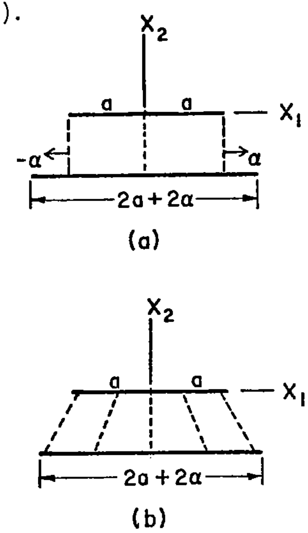

Figure 4
Growth of a crack (a) by translation of crack tips and (b) by similarity transformation

J_2, L AND ENERGY RELEASE RATES

We return to the problem of J_k: does it represent the components of the force acting on a crack, or crack tip, or not? Let us notice that relation (15) is obtained in quite general fashion, for any defect in a continuum, while specific values of J_1 and J_2 given by (2) or (5) were obtained specifically for a plane crack. In this process the nonequivalence of X_1 and X_2 axes is essential. In other words, as long as all directions are equivalent for a defect, J_1 and J_2 have the same meaning of components of a force; however, the geometry of a crack imposes additional constraints to this interpretation.

Let us consider the simplest example of a rigid bar lying on the plane surface axis X_1 being directed along the bar, X_2 being perpendicular to the plane. The force acting at the end of the bar in the X_1 direction causes translation of the whole body in the X_1 direction. However, a force acting in the X_2 direction causes not a translation in the X_2 direction, but a rotation in the X_1X_2 plane about the other end of the bar. The situation for a crack is similar (though not identical): J_1 and J_2 are situated differently with respect to the crack itself, such that their roles are different. To establish the role of J_2, we will pass now to another path-independent integral, namely the L integral defined as

$$L = \oint_C e_{3ij}(WX_jn_i - T_iu_j - T_ku_{k,i}X_j)dS. \quad (17)$$

For the configuration given in Fig. 1,

$$L = 2\sigma_{12}^A(\sigma_{22}^A + \sigma_{11}^A)\pi a^2/E \quad (18)$$

or

$$L = -2aJ_2 + 2\sigma_{12}^A(\sigma_{11}^A - \sigma_{22}^A)\pi a^2/E \quad (19)$$

where J_2 is given by (5). The integral L has been found to be the rotational energy release rate (Ref. 5). The relation has the form:

$$L = -\frac{\partial U}{\partial \Phi}$$

where U is the energy of the crack, depending not only on a, but Φ, the angle by which the crack rotates (virtually) in its plane (Fig. 5).

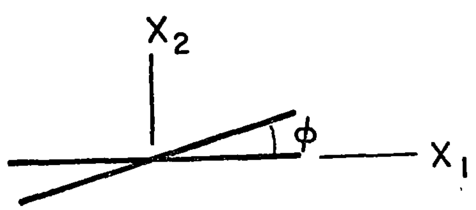

Figure 5

Rotation of Crack

The value of L given by (18) or (19) depends not only on σ_{12}^A and σ_{22}^A but also on σ_{11}^A. From (19) we see that for a certain special stress field L can be expressed in terms of J_2

$$L = -2aJ_2 \quad \text{if} \quad \sigma_{11}^A = \sigma_{22}^A . \quad (20)$$

The relation (20) is similar to (7) except there were no restrictions on the stress field to be satisfied in the case of M and J_1.

The expressions (18) and (20) may be useful in the nondestructive determination of stress intensity factors for cracks subjected to combined loading.

THE CRACK AND GENERALIZED FORCES

The path-dependence of J_2, when taken around one crack tip, indicates that J_2 is not a physically meaningful quantity for measurements concerning properties of materials near a crack. Actually, if we would like to introduce a unified description of the behavior of a crack taking all integrals around the whole crack, we find that

$$J_1 = J_2 = 0$$

and then it is M and L which become physically meaningful. Ascribing to a crack two degrees of freedom, namely extension and rotation, represented by dependence of energy of a crack on a and Φ, we see that L and M are generalized forces associated with those generalized coordinates in the sense of classical mechanics

$$M = -a\frac{\partial U}{\partial a} \quad , \quad L = -\frac{\partial U}{\partial \alpha} .$$

While tensile forces are concentrated only at the tips, the moments are distributed along cracks, too. These two facts explain why M can be expressed through J_1, while L, in general, can not be expressed through J_2.

The next interesting observation can be made in relation to generalized coordinates: if instead of a, Φ we will use generalized coordinates u, γ, as indicated in Fig. 6, we obtain

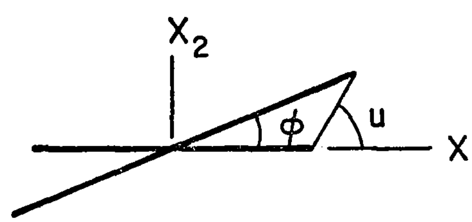

Figure 6

Generalized Coordinates u and Φ

$$\left(\frac{\partial U}{\partial u}\right)_{u-0} = J_1 \cos\gamma - L/a \sin\gamma . \qquad (21)$$

The right-hand side represents a generalized force F_u connected with the generalized coordinate u. For the special case $\sigma^A_{11} - \sigma^A_{22}$ it reduces to

$$F_u = J_1 \cos\gamma + J_2 \sin\gamma . \qquad (22)$$

In this form (Ref. 6) F_u was ascribed to skewing of a crack rather than to rotation. Without assigning this meaning to expression (22), let us notice that if this is true, in a general case we should use (21) involving L rather than (22) which is only a special case of (21).

Finally, we would like to remark briefly on the sign of L. In the definition given by (17), which repeats the well-known (Refs. 3,4) definition of L, the symbol e_{ijk} is used. Should we interchange the indices j and k, the signs in expressions (18),(19) and consequently in (20),(21), would change. As we realize by now, L represents the moment of material forces. Usually, the moment is given by $e_{ijk}\, x_j\, F_k$; it would correspond exactly to the change of sign of L given by (17). Of course, L as given by (17) cannot be represented as a prodict of X_i by another quantity because of the presence of the form $T_i\, u_j$. A more general treatment of path-independent integrals and their relations to conservation laws will be given in (Ref. 7).

ACKNOWLEDGMENT

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SUMMARY DISCUSSION

Otto Buck, Chairman (Rockwell Science Center [now Ames Laboratory]): Any questions, please?

I have a short question. Let's assume we start out in a mode one and the crack goes over into mode two or something like that. Would you be able to treat that particular situation?

George Herrmann (Stanford University): You mean crack skewing? It's not being treated here.

Otto Buck: What hope do you see for that?

George Herrmann: We see some hope, but completely different considerations will be required here.

Perhaps in answering that question, we might mention that what we see here is for the energy releas for rotation, Sigma 1-1. The applied stress parallel to the crack does play a role, and it is expected that in crack skewing, it most likely will therefore also play a role. But we are not prepared to say at this time just what role that will be

Otto Buck: One more question.

Mike Resch (Stanford University): We talked about these conservative integrals for two-dimensional cracks. What does the J integral mean for a three-dimensional, semi elliptical surface crack?

George Herrmann: It means you have to take an integral, not around a line, but you have to take it around the whole surface. In fact, we have some thoughts along that line - or along that surface, I should say, but we have not proceeded any further than just loosely thinking about it.

Chris Burger (Ames Laboratory): Experiments by Kobyashi and Danielson and a few other people seem to suggest to us that the mode two crack; to talk about mode two crack growth, is really not worth much because the crack turns as quickly as it can and propagates in mode one. How does that relate to your work with J-2.

George Herrmann: There are two points of view. There are those people who say that this crack skewing is completely an irrelevant subject because, as you say, the crack just turns right away, and we are not concerned about the details. There are some other people, however, perhaps including some of us, who think that it might be just interesting to see what is going on in detail before the crack is turning and becomes just a pure mode one crack. What is the mechanism which is taking place? Is there any possible additional material constants which govern that process right at the beginning and so on.

Otto Buck, Chairman: Thank you so much, George.